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ITERATED HOMOLOGY AND DECOMPOSITIONS OF SIMPLICIAL COMPLEXES

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ABSTRACT

Kalai has conjectured that a simplicial complex can be partitioned into Boolean algebras at least as roughly as a shifting-preserving collapse sequence of its algebraically shifted complex. In particular, then, a simplicial complex could (conjecturally) be partitioned into Boolean intervals whose sizes are indexed by its iterated Betti numbers, a generalization of ordinary homology Betti numbers. This would imply a long-standing conjecture made (separately) by Garsia and Stanley concerning partitions of Cohen-Macaulay complexes into Boolean intervals.

We prove a relaxation of Kalai's conjecture, showing that a simplicial complex can be partitioned into recursively defined spanning trees of Boolean intervals indexed by its iterated Betti numbers.

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1. Introduction

Iterated homology generalizes ordinary homology, and is related to algebraic shifting, (nonpure) shelling, and depth (see [DR] and Definition 2.3). A simplicial complex Γ of dimension $d-1$ has rth iterated homology groups for each $r = 0, \ldots, d$. When $r = 0$, this corresponds to ordinary homology. If Γ is a cone over Γ', then when $r = 1$, we get the homology of Γ'. The vector space dimensions of the iterated homology groups are the iterated Betti numbers. We will show how the iterated Betti numbers of a simplicial complex index a particular partition of the complex.

Algebraic shifting (see Definition 4.8) is a procedure that transforms a simplicial complex Γ into a new complex $\Delta(\Gamma)$ which is shifted, and hence combinatorially simpler, yet has many of the same combinatorial, algebraic, and topological properties as the original complex. Algebraic shifting was used, for example, by Björner and Kalai [BK] to characterize the f-vectors of simplicial complexes with given Betti numbers. For more recent interpretations of algebraic shifting as a special case of an exterior algebra analogue of generic initial ideals, see [AH] and the references therein.

A shifting-preserving collapse sequence (see Definition 4.3) is a sequence of deletions of Boolean intervals from a simplicial complex, leaving a shifted simplicial complex at each step. The Boolean intervals removed at each step thus partition the complex. Kalai has conjectured that a simplicial complex Γ can be partitioned into Boolean intervals at least as roughly as the partition corresponding to a shifting-preserving collapse sequence of its algebraically shifted complex $\Delta(\Gamma)$:

CONJECTURE 1.1 (Kalai [Ka, Conjecture 7]): *Let Γ be a simplicial complex and let $\Delta(\Gamma) = \biguplus_{i=1}^{t}[R_i, F_i]$ be the representation of the algebraically shifted complex $\Delta(\Gamma)$ as a union of intervals given by a shifting-preserving collapse of $\Delta(\Gamma)$. Then there is a decomposition of Γ into disjoint intervals of the form $\Gamma = \biguplus_{i=1}^{t}[A_i, B_i]$ such that $\dim A_i = \dim R_i$ and $\dim B_i = \dim F_i$.*

For instance, it is a basic result [BK] about algebraic shifting that if Γ is acyclic (has only trivial reduced homology), then $\Delta(\Gamma)$ is a cone over a shifted subcomplex Δ', so $\Delta(\Gamma) = 1 * \Delta'$, where 1 is a particular vertex of $\Delta(\Gamma)$ not in Δ', and $*$ denotes topological join (so $\Delta(\Gamma) = \Delta' \uplus \{F \uplus \{1\} \colon F \in \Delta'\}$). It is not hard to show that, in this case, $\Delta(\Gamma)$ has a shifting-preserving collapse sequence of rank 1 Boolean intervals corresponding to the partition $\Delta(\Gamma) = \biguplus_{F \in \Delta'}[F, F \uplus \{1\}]$. Conjecture 1.1 would then imply that any acyclic simplicial complex can be partitioned into rank 1 Boolean intervals. Stanley [St2] has shown that all acyclic simplicial complexes do have such a partition into rank 1 Boolean intervals.

Even when Γ is an arbitrary (not necessarily acyclic) simplicial complex, $\Delta(\Gamma)$ is at least a near-cone $(1 * \Delta') \uplus B$, where the number of k-dimensional faces in B equals the k-dimensional Betti number of Γ (see [BK] for details). Again, it is not hard to show that $\Delta(\Gamma)$ therefore has a shifting-preserving collapse sequence of rank 0 Boolean intervals, each consisting of one of the faces in B, and rank 1 Boolean intervals of the form $[F, F \uplus \{1\}]$ as before. Conjecture 1.1 would then imply that any simplicial complex can be partitioned into rank 0 and rank 1 Boolean intervals, with the rank 0 intervals indexed by the Betti numbers of the complex. All simplicial complexes do have such a partition into rank 0 and rank 1 Boolean intervals [Du].

We will show in Corollary 4.9 that $\Delta(\Gamma)$ has a canonical shifting-preserving collapse sequence into even larger intervals, indexed by the iterated Betti numbers of Γ. Kalai's conjecture would thus imply that a simplicial complex can be partitioned into Boolean intervals indexed by its iterated Betti numbers (Corollary 4.10). More precisely:

CONJECTURE 1.2: *Let Γ be a simplicial complex. Then there is a decomposition of Γ into disjoint intervals of the form $\Gamma = \biguplus_{i=1}^{t}[A_i, B_i]$ such that $\#\{i\colon \dim A_i = k - r,\ \dim B_i = k\} = \beta^k[r](\Gamma)$, where $\beta^k[r](\Gamma)$ denotes the rth iterated (k-dimensional) Betti number of Γ.*

This conjecture would itself be a significant result, as it would imply a longstanding conjecture due (separately) to Garsia [Ga] and Stanley [St1] that a Cohen-Macaulay simplicial complex can be partitioned into Boolean intervals, whose tops are facets (see Theorem 5.3).

We prove, in Corollary 3.5, a relaxation of Conjecture 1.2. Instead of Boolean intervals, we partition the complex into "Boolean trees" (Definition 3.3), recursively defined spanning trees of Boolean intervals, indexed by the iterated Betti numbers. The main tool is Theorem 3.2, a decomposition of the complex generalizing the one in [Du].

Although the motivation for these results depends upon algebraic shifting and how Conjecture 1.1 implies Conjecture 1.2, we do not need algebraic shifting or either conjecture to prove our main results. We therefore postpone the definitions of shifted complex, shifting-preserving collapse sequence, and algebraic shifting, and the proof that Conjecture 1.1 implies Conjecture 1.2, until Section 4, after the proofs of Theorem 3.2 and Corollary 3.5.

In Section 5, we apply Corollary 3.5 to prove a relaxation of the Garsia-Stanley conjecture, that a Cohen-Macaulay simplicial complex can be partitioned into

Boolean trees, whose tops are facets (Theorem 5.4).

2. Iterated homology

Let Γ be a finite (abstract) simplicial complex. We allow the possibility that Γ is the empty simplicial complex $\emptyset$ consisting of no faces, or the simplicial complex $\{\emptyset\}$ consisting of just the empty face, but we do distinguish between these two cases. The **dimension** of $F \in \Gamma$ is $\dim F = |F| - 1$, and the **dimension** of Γ is $\dim \Gamma = \max\{\dim F\colon F \in \Gamma\}$. The maximal faces of Γ are called **facets**, and Γ is **pure** if all the facets have the same dimension. Let Γ_k denote the set of k-faces (i.e., k-dimensional faces) of Γ. The **f-vector** of Γ is the sequence $(f_0, \ldots, f_{d-1})$, where $f_k = \#\Gamma_k$ and $d-1 = \dim(\Gamma)$. The same notion of $f_k(\Gamma)$ and the f-vector will apply to every finite collection of sets.

Fix a field K throughout the rest of the paper. We call $\beta_i(\Gamma) = \dim_K \tilde{H}^i(\Gamma; K)$ the i**th reduced Betti number** of Γ with respect to K, where $\tilde{H}^i(\Gamma; K)$ is the ith reduced cohomology group with respect to i. The **Betti sequence** of Γ is $\beta(\Gamma) = (\beta_0, \ldots, \beta_{d-1})$. Recall that over a field $\dim_K \tilde{H}^i(\Gamma; K) = \dim_K \tilde{H}_i(\Gamma; K)$, so that the Betti sequence measures reduced homology as well as reduced cohomology of Γ.

Definition 2.1: Let Γ be a $(d-1)$-dimensional simplicial complex with vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. Let $\Lambda(KV)$ denote the exterior algebra of the vector space KV; it has a K-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $S = \{e_{i_1} < \cdots < e_{i_k}\} \subseteq V$ (and $e_\emptyset = 1$). Note that $\Lambda(KV) = \bigoplus_{k=0}^{n} \Lambda^k(KV)$ is a graded K-algebra, and that $\Lambda^k(KV)$ has basis $\{e_S\colon |S| = k\}$.

Let $(I_\Gamma)_k$ be the subspace of $\Lambda^{k+1}(KV)$ generated by $\{e_S\colon |S| = k+1,\ S \notin \Gamma\}$. Then $I_\Gamma := \bigoplus_{k=-1}^{d-1} (I_\Gamma)_k$ is the homogeneous graded ideal of $\Lambda(KV)$ generated by $\{e_S\colon S \notin \Gamma\}$. Let

$$C^k(\Gamma) := \Lambda^{k+1}(KV)/(I_\Gamma)_k.$$

Then the graded quotient algebra

$$\Lambda[\Gamma] := \bigoplus_{k=-1}^{d-1} C^k(\Gamma) = \Lambda(KV)/I_\Gamma$$

is called the **exterior face ring** of Γ (over K).

The exterior face ring is just the exterior algebra analogue of the Stanley-Reisner face ring of a simplicial complex (e.g., [St3]). For $x \in KV$, let $\tilde{x}$ denote the image of x in $\Lambda[\Gamma]$. The set of all **face-monomials** $\{\tilde{e}_S\colon S \in \Gamma\}$ is a K-vector space basis for $\Lambda[\Gamma]$, so $f_k(\Gamma) = \dim_K(C^k(\Gamma))$.

We can use the exterior face ring to compute cohomology. If $f = \alpha_1 e_1 + \cdots + \alpha_n e_n$, then $\delta_f\colon \Lambda[\Gamma] \to \Lambda[\Gamma]$ defined by $\delta_f(x) = \tilde{f} \wedge x$ is a **weighted coboundary operator**, so-called because

$$\delta_f(\tilde{e}_S) = \tilde{f} \wedge \tilde{e}_S = \sum_{i=1}^{n} \alpha_i \tilde{e}_i \wedge \tilde{e}_S = \sum_{\substack{i \notin S \\ S \cup \{i\} \in \Gamma}} \pm \alpha_i \tilde{e}_{S \cup \{i\}}.$$

Setting every $\alpha_i = 1$ gives the usual coboundary operator. Ordinary Betti numbers may be computed using weighted coboundary operators as follows: $\beta_{k-1}(\Gamma) = \dim_K(\ker \delta_f)_{k-1}/(\operatorname{im} \delta_f)_{k-1}$, if $f = \alpha_1 e_1 + \cdots + \alpha_n e_n$ and every α_i is non-zero [BK, pp. 289–290].

Definition 2.2: Let $\{f_1, \ldots, f_n\}$ be a "generic" basis of KV, i.e., $f_i = \sum_{j=1}^{n} \alpha_{ij} e_j$, where the α_{ij}'s are n^2 transcendentals, algebraically independent over K. The **generic coboundary operators** of Γ, $\delta_i\colon \Lambda[\Gamma] \to \Lambda[\Gamma]$, are defined by

$$\delta_i y = \tilde{f}_i \wedge y$$

for $1 \le i \le n$. Furthermore, let $\delta_{(0)} = id$ and, for $r \ge 1$, $\delta_{(r)} = \delta_r \cdots \delta_1$.

Generic coboundaries were introduced by Kalai [Ka], though we are here using the version in [DR]. Note that $\delta_i \delta_j y = -\delta_j \delta_i y$, and so $\delta_{(i)} \delta_{i+1} y = \pm \delta_{i+1} \delta_{(i)} y$.

Definition 2.3: [DR, Section 4]. If Γ is a simplicial complex and $0 \le r \le k+1 \le d$, let

$$\begin{aligned}
C^k[r](\Gamma) &= \delta_{(r)}\left(C^{k-r}(\Gamma)\right),\\
Z^k[r](\Gamma) &= \{x \in C^k[r](\Gamma)\colon \delta_{r+1} x = 0\},\\
B^k[r](\Gamma) &= \begin{cases} \delta_{r+1}\left(C^{k-1}[r](\Gamma)\right) & \text{if } r < k+1 \\ 0 & \text{if } r = k+1 \end{cases},\\
H^k[r](\Gamma) &= Z^k[r](\Gamma)/B^k[r](\Gamma).
\end{aligned}$$

The $H^k[r](\Gamma)$'s are called the r**th iterated cohomology groups** of Γ. The r**th iterated Betti numbers** are defined by

$$\beta^k[r](\Gamma) = \dim H^k[r](\Gamma).$$

The $r = 0$ case is just ordinary reduced cohomology.

For further details about iterated homology see [DR].

A **Betti set** B of a simplicial complex Γ is a set of faces that "count" the reduced Betti numbers of Γ; i.e., $f_k(B) = \beta_k(\Gamma)$ for all k. We extend this concept to iterated homology.

Definition 2.4: A set B of faces of a simplicial complex Γ is an r-**Betti set** of Γ if $f_{k-r}(B) = \beta^k[r](\Gamma)$ for all i.

Note that a 0-Betti set is just a Betti set.

3. Decompositions

In this section, we prove the existence of a key decomposition (Theorem 3.2), and then easily derive our main result (Corollary 3.5). For a given set A, we will let $\theta A := \{\theta a: a \in A\}$ if θ is a function whose domain contains A, and we will let KA denote the K-vector space with basis A.

The following lemma generalizes [Du, Lemma 3.1].

LEMMA 3.1: *Let $\mathcal{G}$ be a directed graph on the n-element vertex set X. Let $\theta: KX \to V$ be an injective linear transformation into some K-vector space V. Suppose that there is a linear transformation $\phi: K(\theta X) \to K(\theta X)$ satisfying the two properties:*

1. *If $x \in X$, then*

$$(1) \qquad \phi(\theta x) \in span_K\{\theta y: y \in X, (x, y) \text{ is an edge of } \mathcal{G}\};$$

and

2. $\operatorname{im}\phi \subseteq \ker\phi$.

Further suppose Z and Y are subsets of X, such that:

3. *$\phi(\theta Z)$ is a basis of $\operatorname{im}\phi$; and*
4. *θY is a basis of $K(\theta X)/\operatorname{im}\phi$.*

Then there is a matching between Z and $W = X - Y$ in $\mathcal{G}$.

Proof: Since $\phi(\theta Z)$ is a basis of $\operatorname{im}\phi$, it follows that $\dim(\operatorname{im}\phi) = |\theta Z|$. Also $|\theta Y| = |\theta X| - \dim(\operatorname{im}\phi)$, so $\dim(\operatorname{im}\phi) = |\theta X| - |\theta Y|$. By injectivity of θ then, $|Z| = |\theta Z| = \dim(\operatorname{im}\phi) = |\theta X| - |\theta Y| = |X| - |Y| = |W|$.

By the Marriage Theorem (e.g., [Ry, Ch. 5, Thm. 1.1]), it suffices to show that for any $Z' \subseteq Z$ with, say, $|Z'| = k$, there are at least k vertices $w_1, \ldots, w_k \in W$ such that for each $1 \le i \le k$ there is a $z \in Z'$ with (z, w_i) an edge of $\mathcal{G}$. Suppose not. Let $Z' = \{z_1, \ldots, z_k\}$. Then $\phi(\theta z_1), \ldots, \phi(\theta z_k)$ are linearly dependent in

$K(\theta X)/K(\theta Y)$, since they are all in the span of fewer than k vectors of $\theta W = \theta X - \theta Y$. Thus there is a linear combination of $\phi(\theta z_1), \dots, \phi(\theta z_k)$ in $K(\theta Y)$, say

$$a_1\phi(\theta z_1) + \cdots + a_k\phi(\theta z_k) = \theta v,$$

where $v \in KY$ and $a_i \in K$, not all $a_i = 0$. Moreover,

$$a_1\phi(\theta z_1) + \cdots + a_k\phi(\theta z_k) = \phi(a_1\theta z_1 + \cdots + a_k\theta z_k) = \theta v \in \operatorname{im}\phi. \tag{2}$$

Since θY is a basis of $K(\theta X)/\operatorname{im}\phi$ and $\theta v \in \operatorname{im}\phi$, we have that $\theta v = 0$. Therefore $a_1\phi(\theta z_1) + \cdots + a_k\phi(\theta z_k) = 0$. Since $\phi(\theta Z)$ is a basis of $\operatorname{im}\phi$, it follows that $a_1 = \cdots = a_k = 0$, a contradiction. ∎

THEOREM 3.2: *Let Γ be a $(d-1)$-dimensional simplicial complex. Then there exists a chain of subcomplexes*

$$\emptyset = \Gamma^{(d+1)} \subseteq \cdots \subseteq \Gamma^{(r)} \subseteq \Gamma^{(r-1)} \subseteq \cdots \subseteq \Gamma^{(1)} \subseteq \Gamma^{(0)} = \Gamma, \tag{3}$$

where

$$\Gamma^{(r)} = \Gamma^{(r+1)} \cup B^{(r)} \cup \Omega^{(r+1)} \quad (0 \le r \le d),$$

and bijections

$$\eta^{(r)}\colon \Gamma^{(r)} \to \Omega^{(r)} \quad (1 \le r \le d), \tag{4}$$

such that, for each r,

1. *$\Gamma^{(r+1)}$ and $\Gamma^{(r+1)} \cup B^{(r)}$ are subcomplexes of $\Gamma^{(r)}$;*
2. *$B^{(r)}$ is an r-Betti set; and*
3. *for any $F \in \Gamma^{(r)}$, we have $F \subseteq \eta^{(r)}(F)$ and $|\eta^{(r)}(F) - F| = 1$.*

Proof: The proof depends upon the generic coboundary operators (Definition 2.2), and proceeds in several steps. Throughout the proof, we adopt the convention that for any subset $\Gamma' \subseteq \Gamma$, we denote by $K\Gamma'$ the K-span within $\Lambda[\Gamma]$ of the images of $\{e_S\colon S \in \Gamma'\}$.

STEP 1: Inductively define $\Gamma^{(r)}$, $0 \le r \le d+1$, so that

$$\delta_{(r)}\Gamma^{(r)} \text{ is a basis for } \delta_{(r)}K\Gamma = K\delta_{(r)}\Gamma. \tag{5}$$

Let $\Gamma^{(0)} = \Gamma$. It is clear that (5) holds for $r = 0$. Now assume that (5) holds for $r-1$; we must find $\Gamma^{(r)}$ satisfying (5). Let

$$I_r = \delta_{(r)}K\Gamma = \delta_r\left(\delta_{(r-1)}K\Gamma\right),$$

a K-vector space. By inductive assumption, I_r is generated by

$$\delta_r\left(\delta_{(r-1)}\Gamma^{(r-1)}\right) = \delta_{(r)}\Gamma^{(r-1)}.$$

Let M_r be the "lexicographically least" basis of I_r; i.e., if $G \in \Gamma^{(r-1)}$ then $\delta_{(r)}x^G \notin M_r$ if and only if

$$\delta_{(r)}x^G = \sum a_i\ \delta_{(r)}\ x^{F_i}, \tag{6}$$

where $a_i \in K$, $F_i <_L G$, and $F_i \in \Gamma^{(r-1)}$. Let

$$\Gamma^{(r)} = \{F \in \Gamma^{(r-1)}\colon \delta_{(r)}\ x^F \in M_r\}.$$

Then it is clear that $\Gamma^{(r)}$ satisfies (5), completing the induction.

Note that $\delta_{(d+1)}$ is the zero map, since its image would have to be at least d-dimensional; therefore (5) guarantees $\Gamma^{(d+1)} = \emptyset$. (A similar dimension argument shows that $\Gamma^{(d)} = \{\emptyset\}$.) It is also easy to see that $\Gamma^{(r)} \subseteq \Gamma^{(r-1)}$, so (3) is satisfied.

STEP 2: Define $B^{(r)}$, $0 \le r \le d$, so that

$$\delta_{(r)}\left(\Gamma^{(r+1)} \cup B^{(r)}\right) \text{ is a basis for } \delta_{(r)}K\Gamma^{(r)}/\delta_{(r+1)}K\Gamma^{(r)}. \tag{7}$$

Let

$$H_r = \delta_{(r)}K\Gamma^{(r)}/\delta_{(r+1)}K\Gamma^{(r)} = K\delta_{(r)}\Gamma^{(r)}/K\delta_{(r+1)}\Gamma^{(r)}.$$

Note that the definition of $\Gamma^{(r)}$ ensures $\delta_{(r)}K\Gamma = \delta_{(r)}K\Gamma^{(r)}$, so

$$\begin{aligned}\delta_{(r+1)}K\Gamma = \delta_{r+1}\delta_{(r)}(K\Gamma) = \delta_{r+1}\left(\delta_{(r)}K\Gamma\right) = \delta_{r+1}\left(\delta_{(r)}K\Gamma^{(r)}\right)\\ = \delta_{(r+1)}K\Gamma^{(r)},\end{aligned}$$

and thus

$$H_r = K\delta_{(r)}\Gamma^{(r)}/K\delta_{(r+1)}\Gamma.$$

Let L_r be the "lexicographically least" basis of H_r; i.e., if $G \in \Gamma^{(r)}$ then $\delta_{(r)}x^G \notin L_r$ if and only if

$$\delta_{(r)}x^G = \sum a_i\ \delta_{(r)}x^{F_i} + \delta_{(r+1)}y, \tag{8}$$

where $a_i \in K$, $F_i <_L G$, $F_i \in \Gamma^{(r)}$, and $y \in K\Gamma$.

We now claim that

$$\Gamma^{(r+1)} \subseteq \{F \in \Gamma^{(r)}\colon \delta_{(r)}x^F \in L_r\}.$$

Assume that $G \in \Gamma^{(r)}$ but $\delta_{(r)}x^G \notin L_r$. By the definition of L_r then, G satisfies (8). Apply δ_{r+1} to both sides of (8). Since $\delta_{r+1}\delta_{(r+1)} = {\delta_{r+1}}^2\delta_{(r)} = 0$, we obtain

$$\delta_{(r+1)}x^G = \sum a_i\, \delta_{(r+1)}\, x^{F_i},$$

so, by the definition of $\Gamma^{(r+1)}$ in Step 1, $G \notin \Gamma^{(r+1)}$. This establishes the claim. We may therefore define

$$B^{(r)} = \{F \in \Gamma^{(r)}\colon \delta_{(r)}x^F \in L_r\} \setminus \Gamma^{(r+1)}.$$

Then it is clear that $\delta_{(r)}\left(\Gamma^{(r+1)} \cup B^{(r)}\right)$ is a basis of H_r, establishing (7).

STEP 3: Show that $\Gamma^{(r+1)}$ and $\Gamma^{(r+1)} \cup B^{(r)}$ are subcomplexes of $\Gamma^{(r)}$.

To show that $\Gamma^{(r+1)}$ is a subcomplex, suppose that $G \subset F$ and $G \in \Gamma^{(r)} - \Gamma^{(r+1)}$. We must show that $F \notin \Gamma^{(r+1)}$. By the definition of $\Gamma^{(r+1)}$ in Step 1, we have that $\delta_{(r+1)}x^G \notin M_{r+1}$, so equation (6) holds (with an index shift). Multiply (the index-shifted) equation (6) on the left by x^{F-G}:

$$\pm\delta_{(r+1)}x^F = \sum \pm\, a_i\, \delta_{(r+1)}x^{F_i \cup (F-G)}$$

where $F_i \cup (F-G) <_L F$ as before. Hence $\delta_{(r+1)}x^F \notin M_{r+1}$, so $F \notin \Gamma^{(r+1)}$. Thus $\Gamma^{(r+1)}$ is a subcomplex.

Similarly, to show that $\Gamma^{(r+1)} \cup B^{(r)}$ is a subcomplex, suppose that $G \subset F$ and $G \in \Gamma^{(r)} - \left(\Gamma^{(r+1)} \cup B^{(r)}\right)$. We must show that $F \notin \Gamma^{(r+1)} \cup B^{(r)}$. By the definition of $\Gamma^{(r+1)} \cup B^{(r)}$ in Step 2, we have that $\delta_{(r)}x^G \notin L_r$, so equation (8) holds. Multiply equation (8) on the left by x^{F-G}:

$$\pm\delta_{(r)}x^F = \sum \pm\, a_i\, \delta_{(r)}x^{F_i \cup (F-G)} + \delta_{(r+1)}z$$

where $z = \pm x^{F-G}y \in K\Gamma$. Since $F_i \cup (F-G) <_L F$ as before, $\delta_{(r)}x^F \notin L_r$ and so $F \notin \Gamma^{(r+1)} \cup B^{(r)}$. Thus $\Gamma^{(r+1)} \cup B^{(r)}$ is a subcomplex.

STEP 4: Show $B^{(r)}$ is an r-Betti set (for $0 \leq r \leq d$).

To be an r-Betti set, $B^{(r)}$ must count $Z[r](\Gamma)/B[r](\Gamma)$, or, equivalently, $\dim(\ker\delta_{r+1}/\operatorname{im}\delta_{r+1})$, with δ_{r+1} interpreted as acting on the domain $C[r](\Gamma) = \delta_{(r)}K\Gamma$. (For simplicity, we are dropping all k subscripts in this step; all statements extend trivially to the finer grading by k.) Note also, as in Step 2, that the definition of $\Gamma^{(r)}$ ensures that $\delta_{(r)}K\Gamma = \delta_{(r)}K\Gamma^{(r)}$.

Now, by (5), the definition of $\Gamma^{(r)}$,

$$|\Gamma^{(r+1)}| = \dim\delta_{r+1}(\delta_{(r)}K\Gamma) = \dim(\delta_{(r)}K\Gamma) - \dim(\ker\delta_{r+1}).$$

Similarly, by (7), the definition of $B^{(r)}$,

$$\begin{aligned}|\Gamma^{(r+1)} \cup B^{(r)}| &= \dim(\delta_{(r)} K\Gamma^{(r)})/\delta_{r+1}(\delta_{(r)} K\Gamma^{(r)}) \\ &= \dim(\delta_{(r)} K\Gamma)/\delta_{r+1}(\delta_{(r)} K\Gamma) \\ &= \dim(\delta_{(r)} K\Gamma) - \dim(\operatorname{im} \delta_{r+1}).\end{aligned}$$

Finally, then

$$|B^{(r)}| = |\Gamma^{(r+1)} \cup B^{(r)}| - |\Gamma^{(r+1)}| = \dim(\ker \delta_{r+1}) - \dim(\operatorname{im} \delta_{r+1})$$

as desired.

STEP 5: Verify the existence of $\eta^{(r)}$, relying upon Lemma 3.1.

Let $\mathcal{G}$ be the directed graph whose vertices are $X = \Gamma^{(r-1)}$ and whose edges are the pairs (F, G) with $F \subset G \in \Gamma^{(r-1)}$ and $|G - F| = 1$. Let $V = K\Gamma$. Define $\theta\colon \Gamma^{(r-1)} \to K\Gamma$ by

$$\theta(F) = \delta_{(r-1)} x^F.$$

By (5), $\theta = \delta_{(r-1)}$ is injective on its domain $K\Gamma^{(r-1)} = KX$. Also define

$$\phi\colon K\delta_{(r-1)}\Gamma^{(r-1)} \to K\Gamma$$

by

$$\phi = \delta_r.$$

Finally, let

$$Z = \Gamma^{(r)}$$

and

$$Y = \Gamma^{(r)} \cup B^{(r-1)}.$$

It remains to show that conditions 1 through 4 of Lemma 3.1 are satisfied.

Condition 2 of the lemma is obvious because $\phi = \delta_r$ is a coboundary operator.

To verify condition 1 of the lemma, let $F \in \Gamma^{(r-1)}$. Define

$$\mathcal{S}_F = \left\{G \in \Gamma^{(r-1)} \colon F \subset G,\ |G - F| = 1\right\}.$$

Because $\phi = \delta_r$ is a coboundary operator, it follows that

$$\begin{aligned}\phi(\theta F) &= \delta_r \left(\delta_{(r-1)} x^F\right) = \pm \delta_{(r-1)} \left(\delta_r x^F\right) \\ &= \pm\, \delta_{(r-1)} \sum_{G \in \mathcal{S}_F} a_G\, x^G \quad (\text{where } a_G \in K) \\ &= \sum_{G \in \mathcal{S}_F} \pm a_G \left(\delta_{(r-1)} x^G\right) \\ &= \sum_{G \in \mathcal{S}_F} \pm a_G\, \theta G,\end{aligned}$$

and so

$$\phi(\theta F) \in \mathrm{span}_K\{\theta G\colon (F,G) \text{ is an edge of } \mathcal{G}\}.$$

To verify condition 3 of the lemma, we need to show that

$$\phi(\theta Z) = \delta_r\delta_{(r-1)}\Gamma^{(r)} = \delta_{(r)}\Gamma^{(r)}$$

is a basis for

$$\begin{aligned}\phi(K(\theta X)) =&\delta_r\left(K\delta_{(r-1)}\Gamma^{(r-1)}\right) = K\delta_r\delta_{(r-1)}\Gamma^{(r-1)}\\ =&K\delta_{(r)}\Gamma^{(r-1)}.\end{aligned}$$

But by (5), $\delta_{(r)}\Gamma^{(r)}$ is a basis for $K\delta_{(r)}\Gamma \supseteq K\delta_{(r)}\Gamma^{(r-1)}$, and $\delta_{(r)}\Gamma^{(r)} \subseteq K\delta_{(r)}\Gamma^{(r-1)}$ (since $\Gamma^{(r)} \subseteq \Gamma^{(r-1)}$), so $\delta_r\Gamma^{(r)}$ is a basis of $K\delta_{(r)}\Gamma^{(r-1)}$.

Finally, to verify condition 4 of the lemma, note that

$$\theta(Y) = \delta_{(r-1)}\left(\Gamma^{(r)} \cup B^{(r-1)}\right)$$

is a basis for

$$\begin{aligned}K\delta_{(r-1)}\Gamma^{(r-1)}/K\delta_{(r)}\Gamma^{(r-1)} =&K\delta_{(r-1)}\Gamma^{(r-1)}/\delta_rK\delta_{(r-1)}\Gamma^{(r-1)}\\ =&K\theta X/\phi(K\theta X),\end{aligned}$$

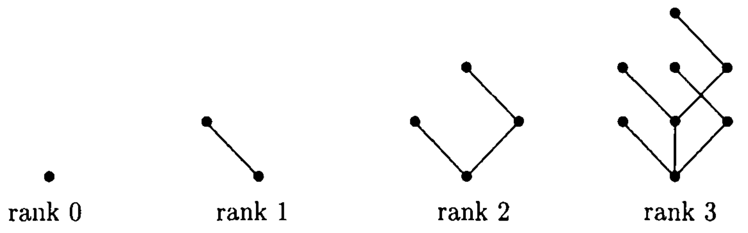
by (7), the definition of $B^{(r-1)}$. ∎

We now use Theorem 3.2 to prove our main result, Corollary 3.5, which differs from Conjecture 1.2 only in that rank r Boolean intervals are replaced by rank r Boolean trees, defined as follows.

Definition 3.3: A **Boolean tree of rank** i is a subposet of a poset P, with a unique minimal element, defined recursively as follows. Any element $x \in P$ is a Boolean tree of rank 0; clearly x is the minimal element. Now assume T_1 and T_2 are two disjoint Boolean trees of rank $(i-1)$, with minimal elements r_1 and r_2, respectively, such that r_2 covers r_1 in P. Then $T_1 \cup T_2$ is a Boolean tree of rank i; clearly r_1 is its unique minimal element.

Here, a subposet of poset P is a poset whose elements are a subset of the elements of P, and whose order relations are a subset of the order relations of P. All Boolean trees of the same rank are isomorphic as posets.

Example 3.4: The first four Boolean trees are shown below.



Remark: It is easy to see that a Boolean tree of rank i is isomorphic to a spanning tree of a Boolean algebra of rank i.

COROLLARY 3.5: *Let Γ be a simplicial complex. Then there is a decomposition of Γ into disjoint Boolean trees such that the number of Boolean trees of rank r with a $(k-r)$-dimensional minimal element is $\beta^k[r](\Gamma)$, where $\beta^k[r](\Gamma)$ denotes the rth iterated (k-dimensional) Betti number of Γ.*

Proof: The minimal elements of the Boolean trees of rank r in this decomposition will be the faces in $B^{(r)}$; since each $B^{(r)}$ is an r-Betti set, there will be $\beta^k[r](\Gamma)$ trees of rank r with a $(k-r)$-dimensional minimal element. We use the following recursive algorithm to construct the trees.

At every step of the algorithm, Γ will be decomposed into disjoint Boolean trees, but each step will combine some trees from the previous step to make larger trees. At the end of step r, the Boolean trees will have rank at most r, the minimal elements of the trees of rank r will be all the faces in $\Gamma^{(r)}$, and, for $0 \leq i \leq r-1$, the minimal elements of the trees of rank i will be all the faces in $B^{(i)}$.

STEP 0: Every face F in $\Gamma = \Gamma^{(0)}$ forms its own Boolean tree of rank 0.

STEP $r+1$ ($r \geq 0$): We may assume, r steps having been completed, that the minimal elements of the Boolean trees of rank r in the decomposition of Γ are all the faces in $\Gamma^{(r)}$. By Theorem 3.2, $\Gamma^{(r)} = \Gamma^{(r+1)} \cup B^{(r)} \cup \Omega^{(r+1)}$, and, for every $F \in \Gamma^{(r+1)}$, there is an $\eta^{(r+1)}(F) \in \Omega^{(r+1)}$ covering it in Γ. For every $(F, \eta^{(r+1)}(F))$ pair, combine the trees of rank r whose minimal elements are F and $\eta^{(r+1)}(F)$, respectively, into a new tree of rank $r+1$; its minimal element is F. This leaves among the Boolean trees of rank r precisely those whose minimal element is in $B^{(r)}$, and completes step $r+1$.

After the $(d+1)$st step, then, there will be no trees of rank greater than d in the decomposition (since $\Gamma^{(d+1)} = \emptyset$), and, for $0 \leq i \leq d$, the minimal elements of the Boolean trees of rank i will be the faces in $B^{(i)}$, as desired. ∎

4. Shifting

In this section, we show how Conjecture 1.2 is a special case of Conjecture 1.1 (Corollary 4.10). The first step (Theorem 4.7) is to show that a shifted complex has a canonical shifting-preserving collapse sequence. The next step (Corollary 4.9) is to show that for an algebraically shifted complex $\Delta(\Gamma)$, the intervals of this sequence are indexed by the iterated Betti numbers of Γ.

Along the way, we define shifted complex, shifting-preserving collapse sequence, and algebraic shifting.

Definition 4.1: If $S = \{i_1 < \cdots < i_k\}$ and $T = \{j_1 < \cdots < j_k\}$ are k-subsets of integers, then $S \leq_P T$ under the **componentwise partial order** if $i_p \leq j_p$ for all p.

Definition 4.2: A collection $\mathcal{C}$ of k-subsets is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex Δ is **shifted** if Δ_k is shifted for every k.

Definition 4.3 (Kalai [Ka, Section 4])*:* A face R of a simplicial complex Δ is called **free** if it is included in a unique facet F. The empty set is a free face of Δ if and only if Δ is a simplex. (This definition is slightly nonstandard in that we are considering facets themselves to be free.) A **collapse step** is the deletion from Δ of a free face and all faces containing it (i.e., the deletion of the interval $[R, F]$). Performing a collapse step may create new facets. A **collapse sequence** is a sequence of collapse steps that reduce Δ to the empty simplicial complex. A **shifting-preserving collapse sequence** is a collapse sequence that leaves a shifted complex at every step.

PROPOSITION 4.4: *If the interval removed at the ith collapse step of a collapse sequence of a simplicial complex Δ is $[R_i, F_i]$, then $\bigcup_i [R_i, F_i]$ partitions Δ.*

Proof: Because the sequence of deletions of intervals reduces Δ to the empty complex, every face of Δ is in one of the intervals. To show that the intervals do not overlap, assume otherwise; let $G \in [R_i, F_i] \cap [R_j, F_j]$, with $i \neq j$, say $i < j$. Then $R_i \subseteq G \subseteq F_j$, which contradicts R_i being free when $[R_i, F_i]$ is being removed, since $[R_j, F_j]$ is not removed until after $[R_i, F_i]$ is. ∎

In order to describe the canonical shifting-preserving collapse sequence in Theorem 4.7, we need the following two definitions. Recall that $[r] = \{1, \ldots, r\}$ if $r \geq 1$, and that $[0] = \emptyset$.

Definition 4.5: Let F be a set of positive integers. Define

$$\operatorname{init}(F) = \min\{r\colon r \notin F\} - 1.$$

Equivalently,

$$\operatorname{init}(F) = \max\{r\colon [r] \subseteq F\}.$$

In other words, $\operatorname{init}(F)$ measures the largest "initial segment" in F, and is 0 if there is no initial segment (i.e., $1 \notin F$).

Definition 4.6: If $S = \{i_1 < \cdots < i_k\}$ and $T = \{j_1 < \cdots < j_m\}$ are sets of integers, then $S \leq_L T$ under the **lexicographic order** if there is a q such that $i_p = j_p$ for $p \leq q$, and either $k = q$ or $i_{q+1} < j_{q+1}$.

Lexicographic order is a total order on all non-empty sets of integers. This definition is more general than the usual one, in that the two sets being compared need not be the same size.

THEOREM 4.7: *If Δ is a shifted simplicial complex, then it has a shifting-preserving collapse sequence whose corresponding decomposition into disjoint Boolean intervals $\Delta = \biguplus_{i=1}^{t}[R_i, F_i]$ satisfies*

$$\#\{i\colon \dim F_i = k,\ \dim R_i = k - r\} = \#\{\text{facets } F \in \Delta\colon \dim F = k,\ \operatorname{init}(F) = r\}. \tag{9}$$

Proof: For every facet F, let

$$R_F := F - [\operatorname{init}(F)].$$

Label the facets of Δ in the opposite of lexicographic order, so

$$F_1 >_L F_2 >_L \cdots >_L F_t$$

are all the facets of Δ. We then specify the collapse sequence by defining the Boolean intervals removed in each collapse step of the collapse sequence to be, in order,

$$[R_{F_1}, F_1], \ldots, [R_{F_t}, F_t]. \tag{10}$$

If (10) is a shifting-preserving collapse sequence, then it is immediate that it satisfies (9). It only remains to show that it is a shifting-preserving collapse sequence.

We first show that the removal of all the intervals in (10) leaves the empty complex, or, equivalently, that every face of Δ is in some $[R_F, F]$. To that end, we first define, for any face $T \in \Delta$,

$$r_T := \max\{r\colon T \cup [r] \in \Delta\}.$$

Clearly,

$$T - [r_T] \subseteq T \subseteq T \cup [r_T]. \tag{11}$$

By definition of r_T, we have $T \cup [r_T] \cup \{r_T + 1\} = T \cup [r_T + 1] \notin \Delta$. Because Δ is shifted, then, $T \cup [r_T] \cup \{v\} \notin \Delta$ for any $v \geq r_T + 1$. Of course, if $v \leq r_T$, then $v \in [r_T] \subseteq T \cup [r_T]$. Therefore, we cannot add any new vertex to $T \cup [r_T]$, and it is thus a facet.

Let $F = T \cup [r_T]$. Then $F \cup \{r_T + 1\} = T \cup [r_T] \cup \{r_T + 1\}$, which is, as above, not in Δ, so $r_T + 1 \notin F$. On the other hand, $[r_T] \subseteq F$, so $\text{init}(F) = r_T$, and (11) may be rewritten as

$$F - \text{init}(F) \subseteq T \subseteq F.$$

This shows that T is in one of the intervals of (10), as desired.

An easy corollary is that no new facets are created as the intervals in (10) are removed. For if T is any face of Δ other than a facet, there is some facet F such that $R_F \subseteq T \subset F$. Then, since F is not removed at least until T is, T is never a facet. We may therefore unambiguously refer to "facets" from now on, without regard to how many of the intervals of (10) have been removed. A further consequence is that if F is a facet, then $[R_F, F]$ is removed when $G \leq_L F$ for all remaining facets G.

Next we show that when facet F is removed, R_F is free. Let G be a facet containing R_F; we must show that $G = F$. Because $G \leq_L F$, either $G \subseteq F$ or $\text{init}(G) \geq \text{init}(F)$; as G is a facet, it must be the latter case. Then $F = [\text{init}(F)] \cup R_F \subseteq [\text{init}(G)] \cup R_F \subseteq G$. Since F is a facet, G must equal F. We conclude that (10) is a collapse sequence.

Finally, to show that (10) is shifting-preserving, we must show that, when $[R_F, F]$ is about to be removed, if $S <_P T$ are two faces still remaining in the complex and $S \in [R_F, F]$, then $T \in [R_F, F]$ as well. Now, $S = R_F \cup I$ for some $I \subseteq [r_F]$. Let T' be the last $|R_F|$ elements of T (i.e., if $T = \{v_1 < \cdots < v_t\}$, then $T' = \{v_{t-|R_F|+1}, v_{t-|R_F|+2}, \ldots, v_t\}$). The set of the last $|R_F|$ elements of $F = \text{init}(F) \cup R_F$ is R_F, so the definition of componentwise partial order $\leq_P$ ensures that $R_F \leq_P T'$. Because Δ is shifted, then, $r_{T'} \leq r_{R_F}$.

Let $G = T' \cup r_{T'}$, a facet. Because F is about to be removed, $G \leq_L F$, so $r_{T'} = r_G \geq r_F = r_{R_F}$. Thus $r_{T'} = r_{R_F}$, and $G \leq_L F$ then implies $T' \leq_L R_F$. But $R_F \leq_P T'$ then implies $T' = R_F$. Therefore $T \supseteq R_F$, and then $T \in [R_F, F]$, since R_F is free. ∎

The following definition relies upon the generic basis $\{f_1, \ldots, f_n\}$ of Definition 2.2.

Definition 4.8: Let Γ be a simplicial complex. Define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_k}$ for $S = \{i_1 < \cdots < i_k\}$ (and set $f_\emptyset = 1$). Define

$$\Delta(\Gamma) := \{S \subseteq [n] \colon \tilde{f}_S \notin \operatorname{span}\{\tilde{f}_R \colon |R| = |S|,\ R <_L S\}\}$$

to be the **algebraically shifted complex** obtained from Γ.

The k-subsets of $\Delta(\Gamma)$ can be chosen by listing all the k-subsets of $[n]$ in lexicographic order and omitting those that are in the span of earlier subsets on the list, modulo I_Γ and with respect to the f-basis. As its name implies, $\Delta(\Gamma)$ is a shifted simplicial complex. For a more complete discussion of algebraic shifting, see [BK].

COROLLARY 4.9: *Let Γ be a simplicial complex, and $\Delta(\Gamma)$ its algebraically shifted complex. Then $\Delta(\Gamma)$ has a shifting-preserving collapse sequence whose corresponding decomposition into disjoint Boolean intervals satisfies*

$$\#\{i \colon \dim F_i = k,\ \dim R_i = k - r\} = \beta^k[r](\Gamma). \tag{12}$$

Proof: Because $\Delta(\Gamma)$ is shifted, it has a shifting-preserving collapse sequence whose corresponding decomposition into disjoint Boolean intervals $\dot\bigcup_{i=1}^t [R_i, F_i]$ satisfies (9), with $\Delta = \Delta(\Gamma)$. But [DR, Theorem 4.1] states that

$$\#\{\text{facets } F \in \Delta(\Gamma) \colon \dim F = k,\ \operatorname{init}(F) = r\} = \beta^k[r](\Gamma). \tag{13}$$

The corollary then follows immediately. ∎

COROLLARY 4.10: *Conjecture 1.1 implies Conjecture 1.2.*

Proof: Let Γ be a simplicial complex, and let $\dot\bigcup_{i=1}^t [R_i, F_i]$ be the decomposition of its algebraically shifted complex $\Delta(\Gamma)$ into disjoint Boolean intervals of $\Delta(\Gamma)$ satisfying (12), guaranteed to exist by Corollary 4.9. If Conjecture 1.1 is true, then there is a decomposition of Γ into disjoint Boolean intervals $\dot\bigcup_{i=1}^t [A_i, B_i]$ such that

$$\#\{i \colon \dim B_i = k,\ \dim A_i = k - r\} = \#\{i \colon \dim F_i = k,\ \dim R_i = k - r\}.$$

But by (12),

$$\#\{i\colon \dim F_i = k, \dim R_i = k - r\} = \beta^k[r](\Gamma).$$

These two equations prove the corollary. ∎

Existing results in the literature suffice to quickly prove almost all of Theorem 4.7 and Corollary 4.9, namely that $\Delta(\Gamma)$ has a collapse sequence, though not necessarily a *shifting-preserving* collapse sequence, whose decomposition satisfies (12). This relies upon the concepts of the h-triangle, a doubly-indexed nonpure generalization of the h-vector, and shellability of nonpure complexes, both due to Björner and Wachs [BW1]. A sketch of a proof along these lines is as follows:

Because $\Delta(\Gamma)$ is shifted, it is shellable [BW2, Corollary 11.4], and therefore has a collapse sequence indexed by its h-triangle [DR, Lemma 5.5]. This is the same collapse sequence that Theorem 4.7 shows to be shifting-preserving. Now, the entries of the h-triangle of $\Delta(\Gamma)$ equal the iterated Betti numbers of Γ [DR, Theorem 5.4]. Thus, $\Delta(\Gamma)$ has a shifting-preserving collapse sequence indexed by the iterated Betti numbers of Γ, which proves Corollary 4.9.

We proved Theorem 4.7 and Corollary 4.9 directly, instead of adding the proof that the collapse sequence is shifting-preserving to the above sketch, because it is easier than introducing the definitions of shellability and the h-triangle.

5. Cohen-Macaulay complexes

In this section, we show how Conjecture 1.2 and Corollary 3.5 apply to Cohen-Macaulay simplicial complexes and a conjecture due (separately) to Garsia [Ga, Remark 5.2] and Stanley [St1, p. 149]. For background on Cohen-Macaulayness, see, e.g., [St3]. We will need to note only that Cohen-Macaulay complexes are pure, and the following result of Kalai's [Ka], recently reproved and generalized by Aramova and Herzog [AH], relating Cohen-Macaulayness and algebraic shifting.

PROPOSITION 5.1 (Kalai): *If Γ is a simplicial complex, then Γ is Cohen-Macaulay if and only if its algebraically shifted complex $\Delta(\Gamma)$ is pure.*

CONJECTURE 5.2 (Garsia, Stanley): *Let Γ be a Cohen-Macaulay simplicial complex. Then there is a decomposition of Γ into disjoint Boolean intervals, whose tops are facets.*

THEOREM 5.3: *Conjecture 1.2 implies Conjecture 5.2.*

Proof: Let Γ be a d-dimensional Cohen-Macaulay simplicial complex. By Proposition 5.1, $\Delta(\Gamma)$ is thus a pure d-dimensional complex. By equation (13), then, for $k < d$,

$$\begin{aligned}\beta^k[r](\Gamma) &= \#\{\text{facets } F \in \Delta(\Gamma)\colon \dim F = k,\ \operatorname{init}(F) = r\} \\ &= 0,\end{aligned}$$

since all of the facets of $\Delta(\Gamma)$ are d-dimensional. If Conjecture 1.2 were true, then Γ could be partitioned into disjoint Boolean intervals $\Gamma = \biguplus_{i=1}^{t}[A_i, B_i]$, where none of the B_i has dimension $k < d$. In other words, Γ could be partitioned into disjoint Boolean intervals, whose tops are facets. ∎

THEOREM 5.4: *If Γ is a Cohen-Macaulay simplicial complex, then Γ can be partitioned into Boolean trees, whose tops are facets.*

Proof: The proof is entirely analogous to that of Theorem 5.3, except using Corollary 3.5 and Boolean trees instead of Conjecture 1.2 and Boolean intervals. ∎

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